MTH 1420, SPRING 2012
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SECTION 5.7: TRIGONOMETRIC SUBSTITUTION AND PARTIAL FRACTIONS

HW: 4, 13, 17, 23, 32

Practice: 5, 11, 15, 18, 21, 25, 29, 33

1. Introduction

As we have seen already, integration is much more complicated than taking derivatives. In this section we look at some additional techniques more integrating certain kinds of integrals. Trigonometric substitution involves making a wholly unintuitive substitution that actually works out a lot of the time, and the method of partial fractions is fun because gives us a trick for integrating rational expressions (plus it gives us a chance to do polynomial division, which is always fun).

2. Trigonometric Substitution

Oftentimes we run into integrals that have factors of the form $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, or $\sqrt{a^2 + x^2}$, where a is some constant (for example, $\sqrt{16 - x^2}$ is of this form). If an extra x term is lying around, we can usually solve these using u substitution. But if there is no x, we can't use substitution. In these cases, it is useful to make a rather strange substitution that allows us to eliminate the square root. The big idea comes from the fact that $\sin^2 x + \cos^2 x = 1$, so if we have something of the form $\sqrt{1 - \cos^2 x}$, we can write it as $\sqrt{\sin^2 x} = \sin x$.

The rule is this:

- If we have a factor of the form $\sqrt{a^2 - x^2}$, we do the substitution $x = a \sin\theta$.
- If we have a factor of the form $\sqrt{a^2 + x^2}$, we do the substitution $x = a \tan\theta$.
- If we have a factor of the form $\sqrt{x^2 - a^2}$, we do the substitution $x = a \sec\theta$.

Example 1. Integrate $\displaystyle\int_{3/2}^{3\sqrt{3}/2} \frac{1}{(\sqrt{9-x^2})^3}\,dx$.

3. Partial Fractions

Consider the integral $\displaystyle\int \frac{x-9}{x^2+3x-10}\,dx$. How could we evaluate this integral?

It seems that none of our previous tricks will work. What we will try to do is use algebra to rewrite the integrand into a form that is easier to integrate. Namely, we want to split it into two fractions that have degree *one* polynomials on the bottom, and constants on the top. Notationally, we want to find numbers A, B, C, D such that

To do this, we will start with the right-hand side of the above equation and combine the fractions. Once we have done that, we can find the appropriate values for A, B, C, D, and then integrate.

The method above is called the method of partial fractions, and all the problems we do in this class will be like the one above. Depending on the denominator, sometimes you have to do other things:

- If one factor is repeated, you need to add in extra terms:
$$\frac{2x}{(x+2)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
- If one factor is an irreducible quadratic, needs $Ax + B$ above it in the partial fraction decomposition.
$$\frac{3x+2}{(x^2+7)(x-3)} = \frac{Ax+B}{x^2+7} + \frac{C}{x-3}$$

We won't be doing any of those types, but you should know that they exist (as do higher-degree analogues). What we *will* do is look at situations where the degree of the polynomial in the numerator is greater than (or equal to) the degree of the polynomial in the denominator. To deal with these, we need to do polynomial long division.

4. Polynomial long division and Partial fractions

Example 2. Integrate $\int \frac{3x^3 + 5x^2 - 12x + 5}{x^2 - x - 6}\,dx.$